# **IV. Elliptic equations**

Our general aim is the consideration of the ***Laplace equation***

 *uxx* + *uyy* = 0. (11.1)

The physical sense of the function *u = u*(*x*,*y*)can be the temperature of the body for the stationary case, the potential of electric field, gravitational field, field of velocities, etc. However, this equation has application in different mathematical directions, for example, in the theory of functions of a complex variable and the variational calculus. This is the subject of this lecture. We will analyze the Laplace equation and its non-homogeneous extension the Poisson equation. Determine, at first, the relation between the solution of the Laplace equation and the analytic functions of complex variable. Then we will talk about the relation between the boundary problem for the Laplace equation and the minimization problem of the Dirichlet integral.

## **11. Elliptic equations. Introductions**

### **11.1. Boundary problems for elliptic equations**

We consider partial differential equations. There are equations with respect to the functions of many variables. The easiest non-trivial case is two-dimensional. For non-stationary systems, there the time and the special variable. These variables are, as a rule independent. Therefore, the considered set, i.e. the domain of the unknown functions is a rectangle. For the two-dimensional case, we have two spatial variables that can be depended.

Consider an equation in a set Ω bounded by a surface *S*. In the one-dimensional case, this surface consists of two points that are the ends of the segment Ω, in the two-dimensional case, this is a closed curve bounding the flat region Ω, and in the three-dimensional case, this is a surface bounding the body Ω.

At each point of the set *S*, a state function value can be specified

*u*(*x*) *= ϕ*(*x*), *x*∈*S*,

where *ϕ* is known function defined on *S*. The equation (11.1) with this condition makes up the ***Dirichlet problem***. The second most important boundary-value problem is the ***Neumann problem***. This corresponds to the boundary condition

 

on the left-hand side of which is the derivative with respect to the external normal to the surface *S*, see Figure 11.1. Depending on the physical meaning of the problem, other types of boundary conditions are possible. In addition, mixed boundary conditions are admissible, for example, a condition of the first type is specified on a part of the boundary, and a second condition on the other.

The described boundary-value problems are called ***internal*** since the processes occurring within the considered region are considered. ***External boundary-value problems*** in which the behavior of the system outside this area is studied are also worthwhile, see Figure 11.1.



Figure 11.1. Internal and external problems.

Now we can have different boundary problem for the Laplace, Poisson and other equations.

### **11.2. Analytical and harmonic functions**

Consider the function *f* of the complex variable *z = x + iy.* This function is ***analytical*** if this is infinite differentiable and can be represented as a Taylor series. Suppose the function *f* is analytic. Then we can calculate its partial derivatives



Therefore, the following equality holds



However, the function *f* has its real part *u* and imagine part *v*, i.e. *f*(*z*)=*u*(*x*,*y*)+*iv*(*x*,*y*). Then we can find



Put the results to the previous equality. We get



This equality can be transformed to



The complex value can be zero whenever its real and imagine part are zero. Then we have two equalities

  (11.1)

These equalities are call the ***Cauchy – Riemann conditions***. The function of complex variable is analytic if and only if it satisfies the ***Cauchy – Riemann conditions***.

Differentiate the first equality (11.1) by *x* and add the second equality (11.1) after its differentiation by *y.* We get

  (11.2)

Now we differentiate the first equality (11.1) by *y* and differ the second equality (11.1) after its differentiation by *x.* We determine

  (11.3)

The relations (11.2) and (11.3) are the Laplace equations. Thus, the function of complex variable is analytic if and only its real and imagine parts satisfy the Laplace equation. The twice-differentiable function of two variable is call harmonic if this is the solution of the Laplace equation. Now we determined that the function of complex variable is analytic if and only its real and imagine parts are harmonic functions.

### **11.3. Minimization of functional for one-dimensional case**

We now the classic result of the function minimization problem. If the differentiable function *f* has a minimum at the point *x*, then its derivative at this point is equal to zero, i.e.

 *f* '(*x*) = 0. (11.4)

Try to use this result for minimization the value



on the set of functions that satisfies the boundary conditions

 *u*(*a*) = 0, *u*(*b*) = 0, (11.5)

where *g* is a given function.

Suppose the function *u* is a solution of this problem. Consider the function

*v*(*x*) = *u*(*x*) + σ*h*(*x*),

where σ is a number, and *h* is an arbitrary function that equal to zero at the boundary.

Determine the function of one variable

*F*(σ) = *I*(*v*) = *I*(*u+*σ*h*).

Note the equality *F*(0) = *I*(*u*). We have the condition *I*(*u*)≤*I*(*v*), because *u* is minimizes the functional *I*. Then for all numbers σ we determine the inequality *F*(0) ≤ *F*(*σ*), i.e. zero is the point of minimum of the function *F.* Therefore, its derivative at this point is equal to zero, by the condition (11.4).

Determine the derivative



Find the value

.

Determine the difference



After division by *σ* and passing to the limit as *σ* tends to zero we get



Thus, we have the equality



Transform first integral, using the formula of integration by parts



because the function *h* is equal to zero at the boundary. By previous formula, we obtain



Note that the function *h* is arbitrary here. Now we use the classic result of the integration theory. If the product of two continuous functions is equal to zero for all value of first function, then the second function is equal to zero. Hence, we determine the equality

  (11.6)

Thus, if the function *u* minimizes the functional *I*, then it satisfies the differential equation (11.6), i.e. for minimization the functional *I* it is necessary to solve the second order differential equation (11.6) with boundary condition (11.5). This is an extension of the condition (11.4) from the minimization problem of function to the minimization problem of functional that depends from the function of one variable.

The function *u*+σ*h* is called ***the variation***, and the method of determining the relation between extremum problem and equations is called the ***variational method***.

### **11.4. Minimization of functional for two-dimensional case**

Extend this result to the functionals that depends from the function of two variables. Consider the functional



that depends from the function of two variables *u = u*(*x*,*y*). This functional is called the ***Dirichlet integral***. It is necessary to minimize it on the class of functions of two variables that is equal to zero on the boundary of rectangle *a*≤*x*≤*b*, *c*≤*y*≤*d.* Try to use the previous technique, i.e. the variational method for solving this problem.

Suppose the function *u* is a solution of this problem. Consider the function

*v*(*x*,*y*) = *u*(*x*,*y*) + σ*h*(*x*,*y*),

where σ is a number, and *h* is an arbitrary function that equal to zero at the boundary.

Determine the function of one variable

*F*(σ) = *I*(*v*) = *I*(*u+*σ*h*).

Note the equality *F*(0) = *I*(*u*). We have the condition *I*(*u*)≤*I*(*v*), because *u* is minimizes the functional *I*. Then for all numbers σ we determine the inequality *F*(0) ≤ *F*(*σ*), i.e. zero is the point of minimum of the function *F.* Therefore, its derivative at this point is equal to zero, by the condition (11.4).

Determine the derivative



Find the value



Determine the difference



After division by *σ* and passing to the limit as *σ* tends to zero we get

.

Integrating by part, we find



because the function *h* is equal to zero on the boundary. Thus, we have the equality

This result is true for all function *h.* We have again the equality to zero of the integral of product of two functions for all values of first of them. Then second multiplier under the integral is equal to zero. Therefore, we obtain

 . (11.7)

Thus, for minimization the functional *I* it is necessary to find the solution of the Laplace equation (11.7) with that equal to zero on the boundary of the considered set. By the way, this is called the ***Dirichlet problem***.

This result is an analogue of the algebraic equation (11.4) and ordinary differential equation (11.6).

Variation method

|  |  |
| --- | --- |
| **extremum problem** | **equation** |
| function of one variable | algebraic equation |
| function of many variables | system of algebraic equations |
| functional of one variable | ordinary differential equation |
| functional of many variables | partial differential equation |

### **Conclusions**

* The general boundary problems for the elliptic equations are the Dirichlet problem (function is given on the boundary) and the Neumann problem (normal derivative is given on the boundary).
* The boundary problems for the elliptic equations can internal or external.
* The Laplace equation has connections with different mathematical directions.
* The function of complex variable is analytic, if its real and imagine and complex part satisfies the Laplace equation.
* Using the variational method, the minimization problem for the Dirichlet integral can be transformed to the Laplace equation.

### Task. **Variational method for the minimization problem**

Consider the problem of minimization of the functional



on the class of functions that equal to zero on the boundary of the given rectangle

Table of parameters

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
| **variant** | ***a*** | ***b*** | ***c*** | ***d*** | ***F*** |
| 1 | 0 | 0 | 1 | 1 |  |
| 2 | -1 | -1 | 0 | 0 |  |
| 3 | 0 | -1 | 1 | 0 |  |
| 4 | -1 | 0 | 0 | 1 |  |
| 5 | 0 | -1 | 2 | 0 |  |
| 6 | -2 | 0 | 0 | 1 |  |
| 7 | -1 | 0 | 1 | 1 |  |
| 8 | 0 | -1 | 1 | 1 |  |

Task: It is necessary to transform the minimization problem to the partial differential equation, using the variational method.